

# Geometry of cuspidal edges with boundary

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We study differential geometric properties of cuspidal edges with boundary. There are several differential geometric invariants which are related with the behavior of the boundary in addition to usual differential geometric invariants of cuspidal edges. We study the relation of these invariants with several other invariants.

## 1 Maps from manifolds with boundary

There are several studies for  $C^\infty$  map-germs  $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$  with  $\mathcal{A}$ -equivalence. Two map-germs  $f, g : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$  are  $\mathcal{A}$ -equivalent if there exist diffeomorphisms  $\varphi : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^m, 0)$  and  $\Phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$  such that  $g \circ \varphi = \Phi \circ f$ . There is also several studies for the case that the source space has a boundary. In [2], map-germs from 2-dimensional manifolds with boundaries into  $\mathbf{R}^2$  are classified, and in [8], map-germs from 3-dimensional manifolds with boundaries into  $\mathbf{R}^2$  are considered. Let  $W \subset (\mathbf{R}^m, 0)$  be a closed submanifold-germ such that  $0 \in \partial W$  and  $\dim W = m$ . We call  $f|_W$  a *map-germ with boundary*, and we call interior points of  $W$  *interior domain of  $f|_W$* . Since  $\partial W$  is an  $(m-1)$ -dimensional submanifold, regarding  $\partial W = B$ , map-germs from manifolds with boundaries can be treated as a map-germ  $f : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$  with a codimension one oriented submanifold  $B \subset (\mathbf{R}^m, 0)$ . We consider  $(\mathbf{R}^m, 0)$  has an orientation and the submanifold  $B$  is considered as the boundary. We define the interior domain of such map-germ  $f$  is the component of  $(\mathbf{R}^m, 0) \setminus B$  such that positively oriented normal vectors of  $B$  points. With this terminology, an equivalent relation for map-germs with boundary is the following. Let  $f, g : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^n, 0)$  be map-germs with codimension one submanifolds  $B, B' \subset (\mathbf{R}^n, 0)$  which contain 0. Then  $f$  and  $g$  are  $\mathcal{B}$ -equivalent if there exist an orientation preserving diffeomorphism  $\varphi : (\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^m, 0)$  such that  $\varphi(B) = B'$ , and a diffeomorphism  $\Phi : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$  satisfies

$$g \circ \varphi = \Phi \circ f.$$

A map-germ  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  is a cuspidal edge if  $f$  is  $\mathcal{A}$ -equivalent to the map-germ  $(u, v) \mapsto (u, v^2, v^3)$  at the origin. We say that  $f$  is a *cuspidal edge with boundary  $B \subset (\mathbf{R}^2, 0)$*  if  $B$  is a codimension one oriented submanifold, that is, there exists a parametrization  $b : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$  to  $B$  satisfying  $b'(0) \neq (0, 0)$ . The domain

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which lies the left hand side of  $b$  with respect to the velocity direction is the interior domain of  $f$ .

In this note, we will consider differential geometric properties of cuspidal edges with boundaries. In order to do this, we first construct a normal form (Proposition 2.1) of it. It can be seen that all the coefficients of the normal form are differential geometric invariants. We give geometric meanings of these invariants. An application of this study is given by considering flat extensions of flat ruled surfaces with boundaries. See [12] for singularities of the flat ruled surfaces, and see [13] for flat extensions of flat ruled surfaces with boundaries. See [3] for flat extensions from general surfaces.

## 2 Normal form of cuspidal edge with boundary

Now we look for normal form of cuspidal edges with boundary. Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  be a cuspidal edge with boundary  $b : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$ ,  $b'(0) \neq (0, 0)$ . One can take a local coordinate system  $(u, v)$  on  $(\mathbf{R}^2, 0)$  and an isometry  $\Phi$  on  $(\mathbf{R}^3, 0)$  satisfying that

$$\Phi \circ f(u, v) = \left( u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3 \right) + h(u, v), \quad (2.1)$$

where  $b_{03} \neq 0$ ,  $b_{20} \geq 0$ , and

$$h(u, v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u, v)),$$

with  $h_1(u), h_2(u), h_3(u), h_4(u), h_5(u, v)$  smooth functions. See [10] for details.

Now we consider  $b$ . Set  $b(t) = (b_1(t), b_2(t))$ . We divide the following two cases.

(1)  $b'_1(0) \neq 0$ ,

(2)  $b'_1(0) = 0$ ,  $b'_2(0) \neq 0$ .

In the case (1), one can take  $u$  for the parameter of  $b$ . Namely,  $b$  is parameterized by

$$b(u) = \left( \varepsilon u, \sum_{k=1}^3 \frac{c_k}{k!} u^k + u^4 c(u) \right) \quad (\varepsilon = \pm 1). \quad (2.2)$$

In the case (2), one can take  $v$  for the parameter of  $b$ . Namely,  $b$  is parameterized by

$$b(v) = \left( \sum_{k=2}^3 \frac{d_k}{k!} v^k + u^4 d(u), \varepsilon v \right) \quad (\varepsilon = \pm 1). \quad (2.3)$$

In summary, we have the following proposition.

**Proposition 2.1.** *For any cuspidal edge  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  with boundary  $b : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$ , there exists a coordinate system on  $(\mathbf{R}^2, 0)$  and an isometry  $\Phi : (\mathbf{R}^3, 0) \rightarrow (\mathbf{R}^3, 0)$  such that  $\Phi \circ f(u, v)$  has the form (2.1) and  $b$  is parameterized by (2.2) (respectively, (2.3)) if  $b'(0) \notin \ker df_0$  (respectively,  $b'(0) \in \ker df_0$ ).*

We remark that all coefficients  $c_1, c_2, c_3, d_2, d_3$  are geometric invariants of cuspidal edge with boundary. Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  be a cuspidal edge. Then there exists a unit vector field  $\nu$  along  $f$  satisfying  $\langle df_p(X), \nu(p) \rangle = 0$  for any  $X \in T_p \mathbf{R}^2$  and  $p \in (\mathbf{R}^2, 0)$ , where  $\langle \cdot, \cdot \rangle$  stands for the Euclidean inner product of  $\mathbf{R}^3$ . We call  $\nu$  *unit normal vector* of  $f$ . Moreover, we see that a couple  $(f, \nu) : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3 \times S^2, (0, \nu(0)))$  is an immersion. Thus a cuspidal edge is a front in the sence of [1]. See also [14].

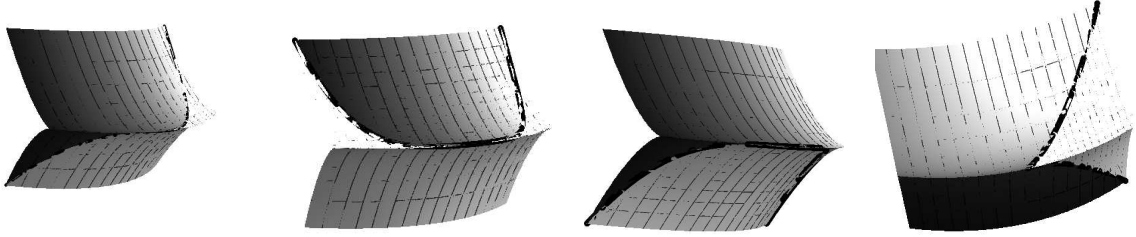


Figure 1: Cuspidal edges with boundary. The boundaries are drawn by thick lines, and the exteriors of the surfaces are drawn by thin colors. Left to right,  $b(t) = (t, t)$ ,  $b(t) = (t, t^2)$ ,  $b(t) = (t, -t^2)$ ,  $b(t) = (t^2, t)$ .

### 3 Differential geometric information

Several geometric invariants on cuspidal edges are defined and studied. See [10, 11, 14] for details. Coefficients of (2.1) are invariants and, according to [10], it is known that  $a_{20}$  coincides with the singular curvature  $\kappa_s$ ,  $b_{20}$  coincides with the limiting normal curvature  $\kappa_\nu$ ,  $b_{03}$  coincides with the cuspidal curvature  $\kappa_c$  and  $b_{12}$  coincides with the cusp-directional torsion  $\kappa_t$  at the origin.

In what follows, we consider the geometry of the boundary. Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  be a cuspidal edge,  $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$  a parametrization of its singular set  $S(f)$ , and  $b : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^2, 0)$  a parametrization of the boundary. We set  $\hat{\gamma}(t) = f \circ \gamma(t)$  and  $\hat{b}(s) = f \circ b(s)$ .

#### 3.1 The case (1)

We assume that  $b'(0) \notin \ker df_0$  and, by this assumption,  $\hat{\gamma} = f \circ \gamma$  and  $\hat{b} = f \circ b$  are both regular curve and they are tangent each other at 0. Hence we have  $l \neq 0$  such that

$$\frac{d}{dt} \hat{\gamma} \Big|_{t=0} = l \frac{d}{ds} \hat{b} \Big|_{s=0}. \quad (3.1)$$

We take a parametrization of  $s$  by  $t$  as  $s = s(t)$ . By the assumption (3.1),  $s'(0) = l$ . Let  $d(t)$  be the curve given by the difference between  $\hat{\gamma}$  and  $\hat{b}$ , that is,

$$d(t) = \hat{\gamma}(t) - \frac{\hat{b}(s(t))}{l}.$$

Then we define the *approaching ratio of boundary to cuspidal edge* (or shortly *approaching ratio*) by

$$\alpha = \left| \frac{1}{|\hat{\gamma}'(0)|^3} \det(\hat{\gamma}'(0), d''(0), \nu(0,0)) \right|^{1/2}, \quad \text{where } ' = \frac{d}{dt},$$

where  $\nu$  is the unit normal vector of  $f$ .

**Lemma 3.1.** *The number  $\alpha$  does not depend on the choice of the parameter  $t$  and the function  $s(t)$ .*

*Proof.* Since

$$\begin{aligned} d'(t) &= \hat{\gamma}'(t) - \frac{1}{l} \frac{d}{ds} \hat{b}(s(t)) s'(t) \\ d''(t) &= \hat{\gamma}''(t) - \frac{1}{l} \left( \frac{d^2}{ds^2} \hat{b}(s(t)) (s'(t))^2 - \frac{d}{ds} \hat{b}(s(t)) s''(t) \right), \end{aligned}$$

and  $(d/ds)\hat{b}|_{s=0}$  is parallel to  $\hat{\gamma}'(0)$ ,  $s'(0) = l$ , we have

$$\alpha = \left| \frac{1}{|\hat{\gamma}'(0)|^3} \det(\hat{\gamma}'(0), \hat{\gamma}''(0) - b_{ss}(0)l, \nu(0,0)) \right|^{1/2}.$$

Thus  $\alpha$  does not depend on  $s(t)$ . We next assume  $t = t(x)$  ( $t(0) = 0$ ) for a parameter  $x$ , and denote  $(\cdot)_x = (d/dx)(\cdot)$ ,  $(\cdot)_s = (d/ds)(\cdot)$ . Then

$$\begin{aligned} & \left| \frac{\det(\hat{\gamma}(t(x))_x, d(t(x))_{xx}, \nu_0)}{|\hat{\gamma}(t(x))_x|^3} \right|_{x=0} \\ &= \left| \frac{\det(\hat{\gamma}'(t(x))t_x(x), \hat{\gamma}''(t(x))(t_x(x))^2 - \hat{b}_{ss}(s(t(x)))(s'(t(x)))^2(t_x(x))^2 l^{-1}, \nu_0)}{|\hat{\gamma}'(t(x))t_x(x)|^3} \right|_{x=0} \\ &= \left| \frac{\det(\hat{\gamma}'(t(x))t_x(x), \hat{\gamma}''(t(x))(t_x(x))^2 - \hat{b}_{ss}(s(t(x)))l(t_x(x))^2, \nu_0)}{|\hat{\gamma}'(t(x))t_x(x)|^3} \right|_{x=0} \\ &= \frac{\det(\hat{\gamma}'(t(0)), \hat{\gamma}''(t(0)) - \hat{b}_{ss}(s(t(0)))l, \nu_0)}{|\hat{\gamma}'(t(0))|^3} \end{aligned}$$

proves the assertion, where  $\nu_0 = \nu(0,0)$ . □

Since the boundary is a curve in  $\mathbf{R}^3$ , its curvature  $\kappa$  and torsion  $\tau$  as a curve in  $\mathbf{R}^3$  are invariants. Moreover,  $\hat{b}$  is a curve on the surface  $f$ . Thus the normal curvature  $\kappa_{nb}$  and the geodesic curvature  $\kappa_{gb}$  of  $b$  are invariants. We have the following proposition for these invariants.

**Proposition 3.2.** *It hold that*

- $\kappa(0) = \sqrt{b_{20}^2 + (c_1^2 + a_{20})^2},$
- $\kappa'(0) = \frac{b_{20}(b_{03}c_1^3 + 3\varepsilon b_{12}c_1^2 + \varepsilon b_{30}) + (c_1^2 + a_{20})(3c_1c_2 + \varepsilon a_{30})}{\sqrt{b_{20}^2 + (c_1^2 + a_{20})^2}},$
- $\tau(0) = \frac{(c_1^2 + a_{20})(\varepsilon b_{03}c_1^3 + 3b_{12}c_1^2 + b_{30}) - b_{20}(3\varepsilon c_1c_2 + a_{30})}{b_{20}^2 + (c_1^2 + a_{20})^2},$
- $\kappa_{nb}(0) = b_{20},$
- $\kappa'_{nb}(0) = \frac{b_{03}c_1^3}{2} + 2\varepsilon b_{12}c_1^2 - \frac{a_{20}b_{03}c_1}{2} + \varepsilon b_{30} - \varepsilon a_{20}b_{12},$
- $\kappa_{gb}(0) = -(\varepsilon c_1^2 + a_{20}),$
- $\kappa'_{gb}(0) = -c_1 \left( \frac{\varepsilon b_{03}b_{20}}{2} + 3\varepsilon c_2 \right) - a_{30} - b_{12}b_{20},$
- $\alpha = |c_1|.$

The invariant  $\alpha$  measures the difference of boundary. We can give a geometric interpretation of  $\alpha$  by using the curvature parabola given by [9] as follows. Let  $f : (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$  be a map-germ satisfying  $\text{rank } df_0 = 1$ , and set

$$N_0f = \{Y \in \mathbf{R}^3; \langle Y, df_0(X) \rangle = 0 \text{ for all } X \in T_0\mathbf{R}^2\},$$

where we identify  $T_0\mathbf{R}^3$  with  $\mathbf{R}^3$ . By this identification,  $N_0f$  is a normal plane of  $df_0(X)$  passing through 0. The curvature parabola  $\Delta_0$  is defined by

$$\Delta_0 = \{a^2 f_{uu}^\perp(0) + 2ab f_{uv}^\perp(0) + b^2 f_{vv}^\perp(0) \in N_0f; a, b \in \mathbf{R}, a^2 E(0) + 2ab F(0) + b^2 G(0) = 1\},$$

where  $E(0) = \langle f_u(0), f_u(0) \rangle$ ,  $F(0) = \langle f_u(0), f_v(0) \rangle$ ,  $G(0) = \langle f_v(0), f_v(0) \rangle$  and, given  $w \in T_0\mathbf{R}^3$ ,  $w^\perp$  is the orthogonal projection of  $w$  at  $N_0f$ . The curvature parabola is a usual parabola if and only if  $f$  is a cross cap, and otherwise,  $\Delta_0$  is a line, a half-line or a point. In [9], the *umbilic curvature* is defined by the distance from the origin to  $\Delta_0$ , if  $\Delta_0$  is a half-line, to the line which contains the half-line. If  $f$  is a cuspidal edge, then  $\Delta_0$  degenerates in a half-line. In this case, the umbilic curvature is equal to the limiting normal curvature defined in [14] up to sign (see also [9, 10]). On the other hand, since  $\hat{b}$  is tangent to  $\hat{\gamma}$  at 0, the principal normal vector  $n$  of  $\hat{b}$  lies in  $N_0f$ . Let  $\ell$  be the line which contains  $\Delta_0$ .

**Lemma 3.3.** *If the limiting normal curvature of the cuspidal edge  $f$  is non zero, then  $0 \notin \ell$ , and  $\ell$  and  $n$  are not parallel.*

*Proof.* Without loss of generality, we can take the normal form for  $f$  as in (2.1). Then after some calculation we get that

$$\Delta_0 = \{(0, a_{20} + t^2, b_{20}); t \in \mathbf{R}\},$$

where the normal plane is  $N_0f = \{(0, y, z); y, z \in \mathbf{R}\}$ . On the other hand,  $n(0) = (0, c_1^2 + a_{20}, b_{20})/\sqrt{(c_1^2 + a_{20})^2 + b_{20}^2}$  which proves the assertion since  $b_{20} \neq 0$ .  $\square$

Let  $V$  be the vertex of  $\Delta_0$ . For instance, for  $f$  given as in (2.1),  $V = (0, a_{20}, b_{20})$ . By Lemma 3.3, if the limiting normal curvature of  $f$  is non zero, there exists a intersection point  $P$  of lines containing  $n$  and  $\ell$ .

**Proposition 3.4.** *If the limiting normal curvature of  $f$  is non zero, then the distance between  $V$  and  $P$  coincides with  $c_1^2$ .*

*Proof.* Like as the proof of Lemma 3.3, we take the normal form for  $f$ . Then  $P = (0, c_1^2 + a_{20}, b_{20})$ , and which proves the assertion.  $\square$

We illustrate the situation in  $N_0f$  of Proposition 3.4 in Figure 2.

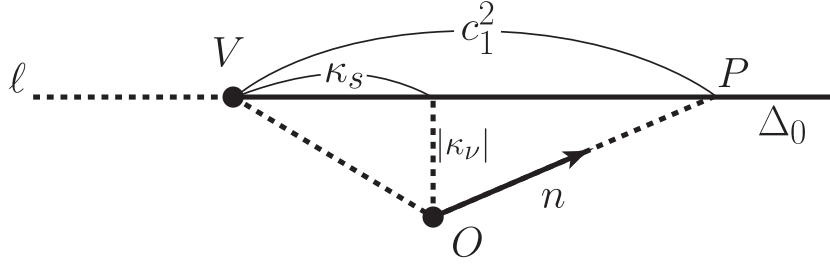


Figure 2: Situation of Proposition 3.4.

### 3.2 The case (2)

We assume that  $b'(0) \in \ker df_0$ , and set  $\hat{b} = f \circ b$ . Then we see that  $\hat{b}'(0) = 0$  and  $\hat{b}''(0) \neq 0$ . Thus we define the *angle between boundary and cuspidal edge* by

$$\beta = \frac{\langle \hat{b}''(0), \gamma'(0) \rangle}{|\hat{b}''(0)| |\gamma'(0)|}.$$

One can easily check that  $\beta$  does not depend on the choice of parameters of  $b$  and  $\gamma$ . If  $f$  is given by the normal form (2.1) with (2.3), we have  $\beta = d_2$ . On the other hand, since  $\hat{b}$  has a singularity, the curvature and torsion may diverge. So we have to prepare curvature and torsion for singular curve. See Appendix A for it. We denote by  $\kappa_{sing}$  (respectively,  $\tau_{sing}$ ) the cuspidal curvature (respectively, the cuspidal torsion) Then the following proposition holds.

**Proposition 3.5.** *The cuspidal curvature and the cuspidal torsion of  $\hat{b}$  satisfies that*

$$\begin{aligned}\kappa_{sing} &= \frac{\sqrt{b_{03}^2(1+d_2^2)+d_3^2}}{(1+d_2^2)^{5/4}}, \\ \tau_{sing} &= \frac{-3\varepsilon a_{20}b_{03}d_2^3 + 3b_{20}d_2^2d_3 + 6b_{12}d_2d_3 - h_5(0,0)d_3 + \varepsilon b_{03}d_4}{(b_{03}^2(1+d_2^2)+d_3^2)^{3/4}}\sqrt{1+d_2^2}.\end{aligned}$$

## 4 Singularities of flat extension of a flat surface

In this section, as an application of the study on cuspidal edges with boundary, we consider flat extensions of a flat ruled surface with boundary. Let  $\gamma : I \rightarrow \mathbf{R}^3$  be a curve satisfying  $\gamma'(t) \neq 0$  for any  $t \in I$ , where  $I$  is an open interval and  $0 \in I$ . Let  $\delta : I \rightarrow S^2$  be a curve satisfying  $\delta'(t) \neq 0$  for any  $t \in I$ , where  $S^2$  is the unit sphere in  $\mathbf{R}^3$ . Then the map  $F : I \times (-\varepsilon, \varepsilon) \rightarrow \mathbf{R}^3$

$$F(t, v) = F_{(\gamma, \delta)}(t, v) = \gamma(t) + v\delta(t), \quad (4.1)$$

where  $\varepsilon > 0$  is called a *ruled surface*. It is known that  $F$  is flat if and only if  $\det(\gamma', \delta, \delta')$  identically vanishes (See [6, Proposition 2.2], for example.). Since  $\delta \neq 0$ , one can assume that the parameter  $t$  is the arc-length. Then  $\{\delta, \delta', \delta \times \delta'\}$  forms an orthonormal frame along  $\delta$ , and

$$\delta''(t) = -\delta(t) + \kappa_\delta(t)\delta(t) \times \delta'(t).$$

The function  $\kappa_\delta$  is called the *geodesic curvature* of  $\delta$ , and  $\delta$  is determined by  $\kappa_\delta$  with an initial condition. On the other hand, we set

$$\gamma'(t) = x(t)\delta(t) + y(t)\delta'(t) + z(t)\delta(t) \times \delta'(t). \quad (4.2)$$

Then  $\gamma$  is determined by  $\{x(t), y(t), z(t)\}$  with an initial condition. Then  $F$  is flat if and only if  $z(t)$  identically vanishes. Moreover, setting  $S(F)$  the singular set of  $F$ , so  $S(F) \cap (I \times [-\varepsilon, \varepsilon]) = \emptyset$  if and only if  $|y| > \varepsilon$  since  $(t, v)$  is a singular point of  $F$  if and only if  $y(t) + u = 0$  as we will see. Thus we set the space of flat ruled surface  $FR$  as

$$FR = \{(x, y, \kappa_\delta) \in C^\infty(I, \mathbf{R} \times (\mathbf{R} \setminus [-\varepsilon, \varepsilon]) \times \mathbf{R})\} \times X,$$

where  $X = \{(\delta_0, \delta_1) \in S^2 \times S^2; \langle \delta_0, \delta_1 \rangle = 0\}$  represents the initial conditions  $\delta(0) = \delta_0$  and  $\delta'(0) = \delta_1$ .

Let us assume that a ruled surface  $F = F_{(\gamma, \delta)}$  satisfies  $S(F) \cap (I \times \{0\}) = \emptyset$ . Then consider extensions of  $F$  for  $v \in (-M, M)$  ( $M > \varepsilon$ ) by the same formula (4.1). We call singular points  $(t, v)$  of  $F$  the *birth of singularities* of extension of  $F$  if  $t$  is a minimal value of  $y(t)$ , since  $(t, v)$  is a singular point of  $F$  if and only if  $y(t) + u = 0$ .

We have the following result.

**Proposition 4.1.** *Let  $I$  be an open interval. Then the set*

$$\mathcal{O} = \{((x, y, \kappa_\delta), (\delta_0, \delta_1)) \in FR ; \text{ all birth of singularities of the extensions of } F_{(\gamma, \delta)} \text{ are cuspidal edges whose } c_1 \text{ vanishes and } c_2 \neq 0\}$$

where  $\gamma$  is defined by (4.2),  $\delta$  is defined by the curvature  $\kappa_\delta$  with the initial condition  $\delta_0, \delta_1$  being open and dense in  $FR$  with respect to the Whitney  $C^\infty$  topology, and  $c_0, c_1$  are given by (2.2).

To prove this proposition, we show the following lemma.

**Lemma 4.2.** *For a flat ruled surface  $F$  as in (4.1),*

- $(t, v)$  is a singular point of  $F$  if and only if  $y(t) + u = 0$ .
- $F$  is a cuspidal edge at  $(t, v) \in S(F)$  if and only if  $y'(t) - x(t) \neq 0$ ,  $\kappa_\delta(t) \neq 0$ .

*Proof.* Since  $F' = \gamma' + u\delta' = x + (y + u)\delta'$  and  $F_u = \delta$ , where we omit  $(t)$  and  $' = \partial/\partial t$ ,  $(\cdot)_u = \partial/\partial u$ , we see the first assertion. Moreover, we see that  $\ker dF_{(t,v)} = \langle \partial t - x\partial u \rangle_{\mathbf{R}}$  for  $(t, v) \in S(F)$ , and  $\delta \times \delta'$  gives a unit normal vector of  $F$ . Set  $\eta = \partial t - x\partial u$ . Thus we see that  $\eta(\delta \times \delta') = \kappa_\delta$ , and  $\eta(y + u) = y' - x$ . By the well-known criteria for cuspidal edge ([15, Corollary 2.5], see also [7, Proposition 1.3]), we see the second assertion.  $\square$

*Proof of Proposition 4.1.* We define subsets of the 2-jet space  $J^2(I, \mathbf{R} \times (\mathbf{R} \setminus [-\varepsilon, \varepsilon]) \times \mathbf{R})$  as follows:

$$\begin{aligned} C_1 &= \{j^2(x, y, \kappa_\delta)(t, v) ; \kappa_\delta(t) = 0\} \\ C_2 &= \{j^2(x, y, \kappa_\delta)(t, v) ; y'(t) - x(t) = 0\} \\ C_3 &= \{j^2(x, y, \kappa_\delta)(t, v) ; y'(t) = 0\} \\ C_4 &= \{j^2(x, y, \kappa_\delta)(t, v) ; y''(t) = 0\} \end{aligned} \tag{4.3}$$

Since a coordinate system of  $J^2(I, \mathbf{R} \times (\mathbf{R} \setminus [-\varepsilon, \varepsilon]) \times \mathbf{R})$  is given by  $(t, x, y, \kappa_\delta, x', y', \kappa'_\delta, x'', y'', \kappa''_\delta)$ , we see that these subsets are closed submanifolds with codimension 1, and  $C_i \cap C_3$  ( $i = 1, 2, 4$ ) are closed submanifolds with codimension 2. By the Thom jet transversality theorem, the set

$$\mathcal{O}' = \{((x, y, \kappa_\delta), (\delta_0, \delta_1)) \in FR ; j^2(x, y, \kappa_\delta) : I \mapsto J^2(I, \mathbf{R} \times (\mathbf{R} \setminus [-\varepsilon, \varepsilon]) \times \mathbf{R}) \text{ is transverse to } C_1, C_2, C_3, C_4 \text{ and } C_i \cap C_3 \text{ (} i = 1, 2, 4 \text{)}\}$$

is a residual subset of  $FR$ . Let  $((x, y, \kappa_\delta), (\delta_0, \delta_1)) \in \mathcal{O}'$  and assume that  $(t_0, v_0)$  is a birth of singularity of  $F$ . Since  $(t_0, v_0)$  is a birth of singularity, and  $S(F) = \{y(t_0) - u_0 = 0\}$ , we see  $y'(t_0) = 0$ . Since  $y'(t_0) = 0$  and  $(x, y, \kappa_\delta) \in \mathcal{O}'$ ,  $F$  at  $(t_0, v_0)$  is a cuspidal edge by Lemma 4.2. Moreover, we have  $y''(t_0) \neq 0$ . This implies that the contact of  $S(F)$  and the  $t$ -curve  $\{(t, v) ; v = v_0\}$  is of second degree. On the other hand, the condition  $c_1 = 0$  and  $c_2 \neq 0$  as in (2.2) implies that the contact of  $S(f)$  (the  $u$ -axis) and  $b$  is of second degree. Since the degrees of contact of two curves do not depend on the diffeomorphism, the cuspidal edge  $F$  at  $(t_0, v_0)$  has the property  $c_1 = 0$  and  $c_2 \neq 0$ . This proves the assertion.  $\square$



We remark that singularities of flat surfaces with boundaries are studied in [12], and the flat extensions of flat surfaces are studied in [13]. Flat extensions of generic surfaces with boundaries are studied in [3]. In [5], flat ruled surfaces approximating regular surfaces are studied.

## A Curvature and torsion of space curves with singularities

In the case (2), the image of the boundary of a cuspidal edge with boundary has a singularity. Thus we need differential geometry of space curves with singularities. In this appendix we give curvature and torsion for space curves with singularities. It should be mentioned that the discussions here are quite analogies of the study for the case of plane curves given by Shiba and Umehara [17], and we follow their discussions in the following.

Let  $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^3, 0)$  be a curve and assume that  $\gamma'(0) = (0, 0, 0)$ . We say that 0 is called *A-type* if  $\gamma''(0) \neq (0, 0, 0)$ , and 0 is called *(2, 3)-type* if  $\gamma''(0) \times \gamma'''(0) \neq (0, 0, 0)$ . Let 0 be a *A-type* singular point of  $\gamma$ , then we define

$$\kappa_{sing} = \frac{|\gamma''(0) \times \gamma'''(0)|}{|\gamma''(0)|^{5/2}}.$$

We call  $\kappa_{sing}$  the *cuspidal curvature* of  $\gamma$ . This definition is analogous to the cuspidal curvature for (2, 3)-cusp of plane curve introduced in [18]. See [16] for detail. Moreover, let 0 be a (2, 3)-type singular point of  $\gamma$ , then we define

$$\tau_{sing} = \frac{\sqrt{|\gamma''(0)|} \det(\gamma''(0), \gamma'''(0), \gamma''''(0))}{|\gamma''(0) \times \gamma'''(0)|^2}.$$

We call  $\tau_{sing}$  the *cuspidal torsion* of  $\gamma$ . By a direct calculation, one can show that  $\kappa_{sing}$  and  $\tau_{sing}$  do not depend on the choice of parameter. Furthermore, we have the following. Let  $s_g$  be the arc-length function  $s_g(t) = \int_0^t |\gamma'(t)| dt$ .

**Fact A.1.** ([17, Theorem 1.1, Lemma 2.1]) *The functions*

$$\operatorname{sgn}(t) \sqrt{|s_g(t)|} \quad \text{and} \quad \sqrt{|s_g(t)|} \kappa(t)$$

*are  $C^\infty$ -differentiable, and*

$$\lim_{t \rightarrow 0} \sqrt{|s_g(t)|} \kappa(t) = \frac{1}{2\sqrt{2}} \kappa_{sing}.$$

By this fact,  $\operatorname{sgn}(t) \sqrt{|s_g(t)|}$  can be taken as a local coordinate of the curve  $\gamma$  at  $t = 0$ . It is called *half-arclength* parameter. We have an analogous claim for the torsion.

**Proposition A.2.** *The function  $\operatorname{sgn}(t)\sqrt{|s_g(t)|}\tau(t)$  is  $C^\infty$  differentiable, and*

$$\lim_{t \rightarrow 0} \operatorname{sgn}(t)\sqrt{|s_g(t)|}\tau(t) = \frac{2}{3\sqrt{2}}\tau_{\text{sing}}.$$

*Proof.* By L'Hôpital's rule, we see

$$\lim_{t \rightarrow 0} \frac{|\gamma' \times \gamma''|^2}{t^4} = \frac{6|\gamma''(0) \times \gamma'''(0)|}{4!}, \quad \lim_{t \rightarrow 0} \frac{\det(\gamma', \gamma'', \gamma''')}{t^3} = \frac{\det(\gamma''(0), \gamma'''(0), \gamma''''(0))}{3!}.$$

Thus these two functions are  $C^\infty$ -differentiable at  $t = 0$ . Moreover,

$$\lim_{t \rightarrow 0} t\tau(t) = \lim_{t \rightarrow 0} \frac{\det(\gamma', \gamma'', \gamma''')}{t^3} \frac{t^4}{|\gamma' \times \gamma''|^2} = \frac{\det(\gamma''(0), \gamma'''(0), \gamma''''(0))}{3!} \frac{4!}{|\gamma''(0) \times \gamma'''(0)|^2}$$

shows that  $t\tau(t)$  is  $C^\infty$ -differentiable. On the other hand, by L'Hôpital's rule, we have

$$\lim_{t \rightarrow 0} \left| \frac{s_g(t)}{t^2} \right| = \lim_{t \rightarrow 0} \frac{|\gamma'(t)|}{|2t|} = \frac{|\gamma''(0)|}{2}. \quad (\text{A.1})$$

Thus

$$\lim_{t \rightarrow 0} \frac{\sqrt{|s_g(t)|}}{|t|} = \frac{\sqrt{|\gamma''(0)|}}{\sqrt{2}}.$$

Hence

$$\begin{aligned} \lim_{t \rightarrow 0} \operatorname{sgn}(t)|t| \frac{\sqrt{|\gamma''(0)|}}{\sqrt{2}} \tau &= \frac{\sqrt{|\gamma''(0)|}}{\sqrt{2}} \lim_{t \rightarrow 0} \frac{\det(\gamma', \gamma'', \gamma''')}{t^3} \frac{t^4}{|\gamma' \times \gamma''|^2} \\ &= \frac{2}{3\sqrt{2}} \frac{\sqrt{|\gamma''(0)|} \det(\gamma''(0), \gamma'''(0), \gamma''''(0))}{|\gamma''(0) \times \gamma'''(0)|^2} \end{aligned}$$

which shows the assertion.  $\square$

We remark that this proof is analogous to that of [17, Lemma 2.1]. Thus  $\kappa_{\text{sing}}$  (respectively,  $\tau_{\text{sing}}$ ) is a geometric invariant of  $A$ -type (respectively,  $(2, 3)$ -type) singular space curve, and it can be regarded as a natural limit of usual curvature (respectively, torsion). We also remark that an  $A$ -type space curve-germ  $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^3, 0)$  at 0 is  $(2, 3)$ -type if and only if  $\kappa_{\text{sing}} \neq 0$ . By (A.1), a parametrization  $t$  of the  $A$ -type space curve-germ  $\gamma$  is the half-arclength parameter if and only if  $|\gamma'(t)| = 2|t|$  (see [17, Remark 2.2]). We have the following proposition.

**Proposition A.3.** *Let  $\alpha, \beta : (\mathbf{R}, 0) \rightarrow \mathbf{R}$  be  $C^\infty$ -functions satisfying  $\alpha > 0$ . Then there exists a unique  $(2, 3)$ -type curve-germ  $\gamma : (\mathbf{R}, 0) \rightarrow (\mathbf{R}^3, 0)$  up to orientation preserving isometric transformations in  $\mathbf{R}^3$  such that*

$$\sqrt{|s_g(t)|}\kappa(t) = \alpha(t) \quad \text{and} \quad \sqrt{|s_g(t)|}\tau(t) = \beta(t) \quad (\text{A.2})$$

and  $t$  is the half-arclength parameter.

*Proof.* Let us consider an ordinary differential equation

$$A'(t) = 2A(t) \begin{pmatrix} 0 & -\alpha(t) & 0 \\ \alpha(t) & 0 & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix}.$$

Then we see that  $A(t)$  is an orthonormal matrix under an initial condition and  $A(0)$  is the identity matrix. Set  $A(t) = (e(t), n(t), b(t))$  and set  $\gamma(t) = 2 \int_0^t te(t) dt$ . Then  $|\gamma'(t)| = 2|t|$  and which shows that  $t$  is the half-arclength parameter. One can easily see that  $\gamma(t)$  satisfies (A.2).  $\square$

We remark that this proof is analogous to that of [17, Theorem 1.1].

For a space curve-germ  $\gamma$  of  $A$ -type, one can easily see that there exist a parameter  $t$  and an isometry  $A$  such that

$$A \circ \gamma(t) = \left( \frac{t^2}{2}, \sum_{i=3}^l \frac{1}{i!} \gamma_{2i} t^i, \sum_{i=4}^l \frac{1}{i!} \gamma_{3i} t^i \right) + (0, O(l+1), O(l+1)), \quad (\text{A.3})$$

where  $O(l+1)$  stands for the terms whose degrees are greater than  $l+1$ , and  $\gamma_{ji} \in \mathbf{R}$  ( $j = 2, 3$ ,  $i = 2, \dots, l$ ). If  $\gamma$  is of  $(2, 3)$ -type, then  $\gamma_{23} \neq 0$ , and we see that

$$\kappa_{sing} = \frac{|\gamma_{23}|}{2\sqrt{2}}, \quad \tau_{sing} = \frac{\gamma_{34}}{\gamma_{23}}.$$

We set

$$\kappa'_{sing} = \frac{d}{dt} \left( \sqrt{|s_g(t)|} \kappa(t) \right) \Big|_{t=0}.$$

Then  $\kappa'_{sing} = (\gamma_{23} + 4\gamma_{24})/(12\sqrt{2}|\gamma_{23}|)$ . Hence we would like to say that  $\kappa_{sing}, \kappa'_{sing}, \tau_{sing}$  are all invariants of  $(2, 3)$ -type singular space curve up to fourth degree. However, it is not easy to compute the differentiation of  $\sqrt{|s_g(t)|} \kappa(t)$  for a given curve. Thus we set

$$\sigma_{sing} = \frac{\left( \langle \gamma''(0) \times \gamma'''(0), \gamma''(0) \times \gamma^{(4)}(0) \rangle - 2 \frac{|\gamma''(0) \times \gamma'''(0)|^2 \langle \gamma''(0), \gamma'''(0) \rangle}{\langle \gamma''(0), \gamma''(0) \rangle} \right)}{\langle \gamma''(0), \gamma''(0) \rangle^{11/4}}.$$

Then this is independent of the choice of the parameter, and

$$\sigma_{sing} = \gamma_{23}(\gamma_{24} - 2\gamma_{23})$$

holds for  $\gamma$  of the form (A.3). Thus invariants  $\{\kappa_{sing}, \sigma_{sing}, \tau_{sing}\}$  can be used instead of  $\{\kappa_{sing}, \kappa'_{sing}, \tau_{sing}\}$  for  $(2, 3)$ -type singular space curve up to fourth degrees.

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